J. Fluid Mech. (1982), *vol.* 119, *pp.* 269–282
Printed in Great Britain

Hamiltonian dynamics of internal waves

By D. MICHAEL MILDER
Aretê Associates, P.O. Box 350, Encino, CA 91316, U.S.A.

(Received 18 May 1981 and in revised form 3 November 1981)

The isopycnal elevation ζ of a stratified incompressible fluid constitutes a generalized co-ordinate in terms of which the dynamics can be described in Hamiltonian form. The description is complete, and no additional variables are necessary, when the isopycnal circulation vanishes and when there are no remote sources of flow. The Hamiltonian, constructed in a co-ordinate system that coincides with the isopycnal surfaces, generates nonlinear equations of motion for ζ and its conjugate variable π which are formally exact for arbitrary displacements of the fluid and which are equivalent at lowest order to the usual linearized equations of internal waves. The appropriate scaling parameters for the nonlinearity are the isopycnal slope $\nabla\zeta$ and vertical strain $\partial_z\zeta$, which emerge as the expansion parameters of an explicit power-series representation of the Hamiltonian.

1. Introduction

Like many other ideal physical systems, an incompressible inviscid fluid obeys Hamilton's principle, yet canonical formulations are rarely applied to fluid problems. One very good reason is that the Lagrangian fluid displacements, with which such a formulation traditionally begins, are in practical terms incompatible with an Eulerian description of the motion. This trouble persists even when density stratification confines the flow to stably oscillating surfaces: Henyey (1981) has demonstrated that while a Hamiltonian formulation of stratified motion is achievable in Eulerian co-ordinates, it must involve explicitly a horizontal Lagrangian displacement variable or some equivalent, 'non-physical', secularly growing quantity.

Nevertheless, Hamiltonian formulations have been used in particular applications. To take advantage of the useful properties of canonical mode equations in the study of weakly nonlinear internal waves, Bretherton & Garrett (1969), and subsequently Meiss, Pomphrey & Watson (1979), have constructed a Hamiltonian around an Eulerian-based expansion of the Lagrangian fluid displacements, which are assumed to remain small. Miles (see his review article; 1981) and others have demonstrated that surface waves on an irrotational fluid can be described canonically in terms of the surface elevation and velocity potential without reference to internal Lagrangian displacements.

When the isopycnal surfaces of an initially quiescent stratified fluid are displaced, a type of wave motion occurs that also admits a complete Hamiltonian formulation in terms of one scalar degree of freedom comprising the surface elevation above equilibrium ζ, and its associated conjugate variable π; such a formulation is the topic of the present paper. The motion consists of 'pure' internal waves for which the circulation in every isopycnal surface remains zero, and for which the flow vanishes at infinity. What makes a simplified Hamiltonian formulation possible in this case is a partially Eulerian description in which the co-ordinate surfaces coincide with the isopycnals; the term in the Hamiltonian containing the secular variable is, in these co-ordinates, proportional to the circulation, and simply vanishes. The remainder of the Hamiltonian consists of a potential energy that is exactly quadratic in ζ, and a kinetic energy that is quadratic in π and which retains additional dependence on the three-dimensional gradient of ζ, a dependence arising from the distortion of the coordinate system.

The residual dependence of the Hamiltonian on the distortion gradient $\partial\zeta$ can in fact be viewed as the origin of the dynamical nonlinearity of the system. The components of $\partial\zeta$, namely the isopycnal slope $\nabla\zeta$ and vertical strain $\partial_z\zeta$, are the ordering parameters of a power-series representation for the Hamiltonian,

$$H[\zeta,\pi] = {}_0H + {}_1H + \ldots, \tag{1.1}$$

in which the leading term ${}_0H$ will be shown to generate the usual linearized equations of motion. The second term, responsible for the principal nonlinear corrections, will be exhibited as a product of $\partial\zeta$ with quadratic physical quantities. From the foregoing it will be possible to conclude that $\nabla\zeta$ and $\partial_z\zeta$ are a sufficient dimensionless measure of the nonlinearity of ideal internal waves, that is, that the linearized equations are valid for $|\partial\zeta| \ll 1$, whatever the ratio of ζ to other length scales describing the medium.

The usual conservation laws for energy and horizontal momentum can be derived directly from Hamilton's principle. The canonical momentum density $-\pi\nabla\zeta$ is quadratic in the field quantities and therefore differs from the local mass-transport rate, but upon volume integration the difference vanishes, as will be seen in § 8.

2. Outline of method

Two properties of ideal internal wave motion in an inviscid, incompressible fluid – conservation of volume and the vanishing of circulation on each isopycnal – will be used to construct a complete kinematical representation of the motion in one scalar variable ζ, the isopycnal elevation, and its time derivative $\dot{\zeta} \equiv \partial\zeta/\partial t$. The dynamical equations for ζ will then be derived from a Langrangian function L, the kinetic energy minus the potential energy,

$$L \equiv T - V, \tag{2.1}$$

through Hamilton's principle,

$$\delta\int L\,dt = 0, \tag{2.2}$$

in the form of the associated Euler–Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta\dot{\zeta}}\right) = \frac{\delta L}{\delta\zeta}. \tag{2.3}$$

The reduction from three velocity components $\mathbf{U}$ to one 'generalized-co-ordinate' velocity $\dot{\zeta}$ comes about through the exploitation of two local differential conditions: the vanishing divergence of $\mathbf{U}$, and the vanishing of vorticity normal to the isopycnal surface. The first condition is an absolute kinematical constraint, while the second is a hybrid condition, consisting of a first integral of the dynamics, supplemented by the initial condition of vanishing circulation. Part of the dynamics (the vanishing of shear stresses) is therefore incorporated *a priori* as an equivalent kinematical constraint, while the remainder is allowed to operate through Hamilton's principle. In § 3 the reduction to one degree of freedom will be carried out in so-called isopycnal co-ordinates, where each fluid element is assigned its horizontal position $\mathbf{x}$ and its *equilibrium* vertical position z_0 as independent variables; the instantaneous elevation ζ relative to z_0 will constitute the dependent variable. The flow constraints take a conveniently simple form in those co-ordinates, and as a consequence the kinetic energy (§§ 3, 6)

$$T = \frac{1}{2}\int \rho U^2 \, d^3x, \tag{2.4}$$

and the gravitational potential energy (§ 4)

$$V = g\int \rho z \, d^3x \tag{2.5}$$

are readily expressed as functionals of ζ and $\dot{\zeta}$. In turn, the nonlinear dynamical equation (2.3), worked out in § 5, has a relatively simple and physically interpretable structure. For these reasons, the isopycnal co-ordinate system seems naturally suited to ideal internal waves, notwithstanding its curvilinear and time-dependent character.

The use of Hamilton's principle deserves some comment. In previous treatments of irrotational surface waves (Miles 1981) its validity has not been taken for granted, but rather proved from the known solutions to the continuum equations. In the present application to internal waves, its validity must be established *a priori*. This can be done as follows (see Lanczos (1966) for a lucid and pertinent account). Like any conservative mechanical system subject to constraints that do no work, an incompressible fluid obeys

$$\delta\int_{t_1}^{t_2} (T-V)\, dt - \int (\rho\mathbf{U} \, . \, \delta\mathbf{X})\, d^3x\Big|_{t_1}^{t_2} = 0, \tag{2.6}$$

for an arbitrary allowed variation $\delta\mathbf{X}$ of the fluid particle trajectories $\mathbf{X}(t)$. Hamilton's principle is valid in a reduced representation $\zeta(t)$ only if setting $\zeta(t_1) = \zeta(t_2) = 0$ causes the second (boundary) term above to vanish. However, the boundary term does not necessarily vanish in incompressible flow, because the volume (and circulation) constraints are non-holonomic, that is they establish a relation between $\dot{\zeta}$ and $\mathbf{U}$ which cannot be integrated to provide a unique connection between ζ and the Lagrangian fluid displacements $\mathbf{X}$. The usual method of dealing with a nonholonomic constraint of the form $\nabla . \mathbf{U} = 0$, for example, is to impose it implicitly through an extra term containing a Lagrange multiplier ϕ,

$$L' = L + \int \rho\phi\nabla . \mathbf{U}\, d^3x, \tag{2.7}$$

so that once ϕ is defined by the condition $\delta' L' = 0$ for those variations $\delta'\mathbf{U}$ that violate the constraint, the displacements $\delta\mathbf{X}$ are formally arbitrary and can be set to zero at t_1 and t_2. (Another term is required for the circulation constraint, but its multiplier can be shown to vanish identically when the circulation itself vanishes.) Apart from details pertaining to isopycnal co-ordinates, this argument constitutes the proof that the modified Lagrangian L' satisfies Hamilton's principle. Curiously, the proof extends to

L itself, for when ϕ and $\mathbf{U}$ are expressed as functions of ζ and $\dot{\zeta}$ by explicit use of the constraints, L' and L are identical for all variations of ζ and $\dot{\zeta}$. This implies that L and L' can be used interchangeably in the dynamical equation (2.3), the only difference being that L' carries extra variables and is subject to additional side conditions.

This argument might usefully be contrasted with Bretherton's application of Hamilton's principle to a perfect compressible fluid (1970). There, the material trajectories are unconstrained to arbitrary infinitesimal variations that vanish at designated initial and final times, so that Hamilton's principle is valid without modification, and can be shown to imply the usual continuum dynamics.

As will be seen in § 5, the extra multiplier ϕ takes the form of a layered potential playing an essential role in the connection between $\mathbf{U}$ and $\dot{\zeta}$. This dual function for ϕ makes L' actually more convenient for deriving the conjugate variable

$$\pi = \frac{\delta L'}{\delta \dot{\zeta}}, \tag{2.8}$$

and for evaluating the nonlinear terms in the dynamical equation for $\dot{\pi} \equiv \partial\pi/\partial t$,

$$\dot{\pi} = \frac{\delta L'}{\delta \zeta}. \tag{2.9}$$

The Hamiltonian form of the dynamical equations,

$$\dot{\zeta} = \frac{\delta H}{\delta \pi}, \quad \dot{\pi} = -\frac{\delta H}{\delta \zeta}, \tag{2.10}$$

is obtained when all quantities in the Hamiltonian functional

$$H[\zeta, \pi] \equiv \int \dot{\zeta}\pi \, d^3x - L \tag{2.11}$$

are expressed in terms of ζ and π. This representation, favoured in the study of weakly nonlinear internal waves and useful in the discussion of conservation laws (see § 8) will be derived out to leading nonlinear order in § 7.

3. Isopycnal co-ordinates

Each fluid element will be labelled by its two-dimensional horizontal position $\mathbf{x}$ and its density ρ, or equivalently, by its equilibrium height z_0, where $\rho(z_0)$ is the equilibrium stratification profile. The co-ordinates $(\mathbf{x}, t, z_0)$ define the isopycnal surface (z_0), whose instantaneous elevation

$$\zeta(\mathbf{x}, t, z_0) \equiv z(\mathbf{x}, t, z_0) - z_0 \tag{3.1}$$

is one of the dependent variables. The partial derivatives $\nabla \equiv \partial/\partial\mathbf{x}$, $\partial_z \equiv \partial/\partial z_0$, and $\partial/\partial t$ are defined at fixed z_0 and follow the surface, so that they differ from their Eulerian counterparts according to

$$\nabla = \nabla_{\mathrm{E}} + (\nabla\zeta)\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t}\right)_{\mathrm{E}} + \dot{\zeta}\frac{\partial}{\partial z}, \tag{3.2a, b}$$

$$\frac{\partial}{\partial z_0} = \left(1 + \frac{\partial \zeta}{\partial z_0}\right)\frac{\partial}{\partial z}. \tag{3.3}$$

These equations need not be referred to again, except for (3.3), in the form

$$d^3x = j(\mathbf{x}, t, z_0)\, dx\, dz_0, \quad j \equiv 1 + \partial\zeta/\partial z_0, \tag{3.4a,b}$$

relating volume to projected isopycnal area dx. The Jacobian j contains the dynamical variable ζ and is just the distorted layer thickness, or one plus the vertical strain. The total time derivative of a quantity makes use of the horizontal (not tangential) velocity $\mathbf{u} = d\mathbf{x}/dt$,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}.\nabla, \tag{3.5}$$

and being an invariant is the same as the ordinary total time derivative, as one can verify from (3.2). The vertical velocity is related to ζ by

$$w \equiv \frac{d\zeta}{dt} = \dot{\zeta} + \mathbf{u}.\nabla\zeta. \tag{3.6}$$

Conservation of volume can be expressed in either of the equivalent forms

$$\nabla.j\mathbf{u} + \partial_z\dot{\zeta} = 0, \tag{3.7a}$$

$$\frac{\partial j}{\partial t} + \nabla.ju = \frac{dj}{dt} + j\nabla.\mathbf{u} = 0. \tag{3.7b}$$

It is worth emphasizing that ρ, being a function only of z_0, satisfies $\partial\rho/\partial t = 0$ and $\nabla\rho = 0$. The vertical gradient of ρ is non-vanishing, but in isopycnal co-ordinates is independent of $\mathbf{x}$ and t, so that

$$g\rho(z_0)\, N^2(z_0) \equiv -\partial_z\rho \tag{3.8}$$

defines a stability frequency N that depends only on z_0.

Isopycnal coordinates have many of the familiar attributes of streamlines and stream surfaces in steady flow. These include conservation of circulation, a property which takes the form

$$C + \Omega\int dx = \text{const.} \tag{3.9}$$

in the presence of rotation at angular rate $\frac{1}{2}\Omega$ around a vertical axis when the circulation contour is on an isopycnal. Here

$$C = \oint (d\mathbf{x}.\mathbf{u} + w\,dz) = \oint d\mathbf{x}.(\mathbf{u} + w\nabla\zeta), \tag{3.10}$$

so that in terms of

$$\tilde{\mathbf{u}} \equiv \mathbf{u} + w\nabla\zeta, \tag{3.11}$$

and

$$\omega \equiv \nabla \times \tilde{\mathbf{u}}, \tag{3.12}$$

the theorem implies

$$\int dx(\omega + \Omega) = \text{const.}, \tag{3.13}$$

which, combined with volume conservation in the form

$$\int j\, dx = \text{const.},$$

gives locally

$$\frac{d\tilde{\omega}}{dt} = 0, \quad \tilde{\omega} \equiv j^{-1}(\omega + \Omega). \tag{3.14}$$

In isopycnal co-ordinates the local conservation of potential vorticity $\tilde{\omega}$ is stated in terms of a layered vorticity ω formed from the isopycnal curl $\nabla\times$, applied to the associated 'dual' velocity $\tilde{\mathbf{u}}$, which, except for a scale factor differing slightly from unity, is the velocity component instantaneously tangent to the surface.

A canonical treatment of the general case requires the introduction of a Lagrange multiplier μ to maintain the constraint (3.14) in a term $\omega\, d\mu/dt$ added to the kinetic-energy density. The variables ω, μ form a canonical pair, but μ is a secularly growing quantity obeying

$$\frac{d\mu}{dt} = \psi, \tag{3.15}$$

where ψ is a 'stream' potential representing the vortical part of the velocity, $\mathbf{u}_r = j^{-1}\hat{\mathbf{z}} \times \nabla\psi$ ($\hat{\mathbf{z}}$ is the unit vertical vector). The variable μ is 'non-physical' in the sense that its values can be altered by the imposition of an arbitrary initial condition $\mu(\mathbf{x}, z_0, t_0)$ without consequence for the fluid trajectories. As Henyey has shown (1981), the general Hamiltonian cannot be constructed without the two variables (ω, μ) or their equivalent, one of which must be 'non-physical'.

However, the special case of vanishing isopycnal vorticity, which can hold when the co-ordinate system is not rotating, can be formulated without the extra variables. In this case the dual velocity $\tilde{\mathbf{u}}$ can be represented on each layer by a single scalar potential ϕ,

$$\tilde{\mathbf{u}} = \nabla\phi, \tag{3.16}$$

and indeed with a little algebra (3.6) and (3.11) can be rewritten in terms of $\nabla\phi$ and $\dot{\zeta}$ as

$$w = [1+(\nabla\zeta)^2]^{-1}(\dot{\zeta}+\tilde{\mathbf{u}}\,.\,\nabla\zeta), \tag{3.17}$$

$$\mathbf{u} = \tilde{\mathbf{u}} - w\nabla\zeta, \tag{3.18}$$

so that once the necessary connection between ϕ and $\dot{\zeta}$ is established by means of the incompressibility equation (3.7a), the kinetic energy can be obtained as a unique functional of $\dot{\zeta}$ and ζ. In the derivation, to be completed in § 6, the following relations will be helpful:

$$u^2 + w^2 = \mathbf{u}\,.\,\tilde{\mathbf{u}} + w\dot{\zeta} \tag{3.19a}$$

$$= \tilde{u}^2 + [1+(\nabla\zeta)^2]^{-1}[\dot{\zeta}^2 - (\tilde{\mathbf{u}}\,.\,\nabla\zeta)^2]. \tag{3.19b}$$

4. Potential energy

The gravitational potential energy per unit horizontal area in a column of fluid extending from z_{b} up to z_{s} at a free surface is

$$g\int_{z_{\mathrm{b}}}^{z_{\mathrm{s}}} z\rho\, dz = \tfrac{1}{2}g\rho z^2\Big|_{z_{\mathrm{b}}}^{z_{\mathrm{s}}} - \tfrac{1}{2}g\int_{\rho_{\mathrm{b}}}^{\rho_{\mathrm{s}}} z^2\, d\rho; \tag{4.1}$$

with isopycnal labelling, $z = z_0+\zeta$, $g\,d\rho = \rho N^2\, dz_0$, this becomes

$$\tfrac{1}{2}g[(z_0^{(\mathrm{s})}+\zeta_{\mathrm{s}})^2 - z_{\mathrm{b}}^2] + \frac{1}{2}\int_{(\mathrm{b})}^{(\mathrm{s})}(z_0+\zeta)^2\rho N^2\, dz_0. \tag{4.2}$$

Terms linear in ζ vanish after integration over x because of volume conservation,

$$\int \zeta\, dx = 0,$$

so that the available potential energy can be recognized as

$$V[\zeta] = \tfrac{1}{2}g\rho_s \int \zeta_s^2 dx + \frac{1}{2}\int \zeta^2 \rho N^2 \, dz_0 \, dx. \tag{4.3}$$

Note that V is exactly quadratic in both surface and internal elevations ζ, with field-independent coefficients ρ_s and ρN^2; the buoyancy co-ordinate forces are purely linear in ζ, and the nonlinear dynamics will have their origin in the kinetic energy.

5. Lagrange's equations of motion

The modified Lagrangian (2.7) has the isopycnal form

$$L'[\mathbf{u}, \zeta, \dot{\zeta}, \phi] = \frac{1}{2}\int [j(u^2 + w^2) + \phi(\nabla . j\mathbf{u} + \partial_z \dot{\zeta}) - N^2\zeta^2] \rho \, dx \, dz_0; \tag{5.1}$$

the multiplier ϕ enforces the compressibility constraint and allows L' to remain stationary to variations of $\mathbf{u}$ that are independent of variations of $\dot{\zeta}$. Variations of w are related to variations of $\mathbf{u}$, ζ and $\dot{\zeta}$ by

$$\delta w = \delta\dot{\zeta} + \nabla\zeta . \delta\mathbf{u} + \mathbf{u} . \nabla\delta\zeta. \tag{5.2}$$

Stationarity of L' to variations of ϕ merely reproduces the compressibility constraint, while stationarity to variations of $\mathbf{u}$, with the help of

$$\begin{aligned} \tfrac{1}{2}j\delta(u^2 + w^2) &= j\mathbf{u} . \delta\mathbf{u} + jw\nabla\zeta . \delta\mathbf{u} \\ &= j\tilde{\mathbf{u}} . \delta\mathbf{u}, \end{aligned} \tag{5.3}$$

and of

$$\phi\delta\nabla . j\mathbf{u} = \nabla . \phi j\delta\mathbf{u} - j\nabla\phi . \delta\mathbf{u}, \tag{5.4}$$

implies

$$j(\tilde{\mathbf{u}} - \nabla\phi) . \delta\mathbf{u} = 0, \tag{5.5}$$

so that ϕ is identical to the layered potential introduced previously to represent $\tilde{\mathbf{u}}$, which is constained to be irrotational.

The conjugate variable $\pi \equiv \delta L/\delta\dot{\zeta}$ can now be evaluated through the variation of L' with ϕ and $\mathbf{u}$ fixed; this is, upon partial integration over z_0,

$$\begin{aligned} \delta L' &= \int [jw\delta\dot{\zeta} + \phi \, \partial_z \delta\dot{\zeta}] \rho \, dx \, dz_0 \\ &= \int (j\rho w - \partial_z \rho\phi) \, \delta\dot{\zeta} \, dx \, dz_s + \rho_s \int \phi_s \delta\dot{\zeta}_s dx, \end{aligned} \tag{5.6}$$

where the boundary term occurring at the upper surface $z_0 = z_s$ has been retained to account for variations that occur when the surface is free, $\delta\dot{\zeta}_s \neq 0$. The conjugate variable is therefore

$$\pi = j\rho w - \partial_z \rho\phi. \tag{5.7}$$

This equation indicates that $\rho\phi$ is vertically continuous, and it establishes a dynamical connection among the values of $\rho\phi$ on different layers. When the upper surface is free its motion is governed by the distinct degree of freedom (ζ_s, π_s), with

$$\pi_s = \rho_s \phi_s, \tag{5.8}$$

a degree of freedom that incorporates the additional surface potential energy term in (4.3). The equation above resembles the known result for surface waves on an irrotational fluid (Miles 1981), except that the surface velocity potential has been replaced here by a stratified potential.

The variational derivative necessary to complete the equation of motion is $\delta L/\delta\zeta$, which is the same as $\delta L'/\delta\zeta$ with $\mathbf{u}$, $\dot{\zeta}$, and ϕ fixed. The variable terms are

$$\delta[\tfrac{1}{2}j(u^2+w^2)] = \tfrac{1}{2}(u^2+w^2)\,\delta j + jw\mathbf{u}\,.\,\nabla\delta\zeta \tag{5.9}$$

(see (5.2)),

$$\delta(\phi\nabla\,.\,j\mathbf{u}) = \phi\nabla\,.\,(\mathbf{u}\delta j), \tag{5.10}$$

$$-\delta(\tfrac{1}{2}N^2\zeta^2) = -N^2\zeta\delta\zeta, \tag{5.11}$$

where

$$\delta j = \partial_z\,\delta\zeta,$$

and after partial integration these yield

$$\frac{\delta L}{\delta\zeta} = -\rho[N^2\zeta + \nabla\,.\,j\mathbf{u}w] + \partial_z\rho[\mathbf{u}\,.\,\nabla\phi - \tfrac{1}{2}(u^2+w^2)]. \tag{5.12}$$

The corresponding equation of motion is then

$$\dot{\pi} + \rho N^2\zeta = -\rho\nabla\,.\,j\mathbf{u}w + \tfrac{1}{2}\,\partial_z\rho\{(\nabla\phi)^2 - [1+(\nabla\zeta)^2]\,w^2\}. \tag{5.13}$$

Here the nonlinear terms have been put on the right-hand side, and the last term has been reworked into symmetrical form with the help of (3.6), (3.17) and (3.19).

Vertical gradients of inertial density, insofar as these may be important, occur in the last term above, and are conveniently measured by an inverse Boussinesq length scale

$$b(z_0) \equiv -\rho^{-1}\partial_z\rho = N^2/g. \tag{5.14a, b}$$

Direct evaluation of the term in question yields

$$b\rho[\tfrac{1}{2}(u^2+w^2) - \mathbf{u}\,.\,\nabla\phi] + \rho w\,dj/dt - \rho\mathbf{u}\,.\,\nabla\,\partial_z\phi \tag{5.15}$$

(see (3.7), (3.16), (3.18)), which can be combined with

$$\mathbf{u}\,.\,\nabla\pi - \nabla\,.\,j\rho\mathbf{u}w = \rho w\frac{dj}{dt} - \rho\mathbf{u}\,.\,\nabla(\partial_z - b)\,\phi, \tag{5.16}$$

derivable from (3.7) and (5.7), to produce an unexpectedly simple variant of the equation of motion,

$$\frac{d\pi}{dt} + \rho N^2\zeta = \tfrac{1}{2}b\rho(u^2+w^2). \tag{5.17}$$

In the Boussinesq approximation, $b = 0$, the production rate of conjugate momentum π following a material point is proportional to minus the elevation. Because the term accompanying b is quadratic in the flow quantities, the linearized equation automatically obeys the Boussinesq approximation; however, as will be seen in § 6, a term containing b survives in the linearized approximation for π as a functional of $\dot{\zeta}$.

6. Relations among the kinetic variables

The modified Lagrangian yields the exact equations of motion (5.7), (5.13) without recourse to a complete representation of L as a functional of ζ and $\dot{\zeta}$. The equations are not formally complete, however, until the quantities ϕ, $\mathbf{u}$, and w appearing in (5.13) can be specified as functionals of $\dot{\zeta}$. The necessary relation is supplied by the incompressibility equation (3.7a), $\nabla\,.\,j\mathbf{u} + \partial_z\dot{\zeta} = 0$, applied to (3.18) in the form

$$j\mathbf{u} = j\tilde{\mathbf{u}} - \eta(\dot{\zeta} + \tilde{\mathbf{u}}\,.\,\nabla\zeta)\,\nabla\zeta, \tag{6.1}$$

where η is an abbreviation for

$$\eta \equiv j[1+(\nabla\zeta)^2]^{-1}; \tag{6.2}$$

substitution into (3.7a) then yields

$$\nabla_*^2 \phi + d_z^* \dot{\zeta} = 0. \tag{6.3}$$

Here the two-dimensional 'Laplacian' operator is defined by

$$\nabla_*^2 \equiv \nabla . j\nabla - \nabla . (\eta\nabla\zeta)(\nabla\zeta) . \nabla, \tag{6.4}$$

where ∇ is understood to operate on everything to its right unless contained within parentheses. The operator d_z^* and the related operator d_z are defined as

$$d_z \equiv \partial_z - \eta(\nabla\zeta) . \nabla, \tag{6.5a}$$

$$d_z^* \equiv \partial_z - \nabla . (\eta\nabla\zeta), \tag{6.5b}$$

expressions that are mutually anti-adjoint under volume integration in the sense that

$$\int (f d_z g + g d_z^* f)\, dx\, dz_0 = \int (fg)\, dx \Big|_{z_b}^{z_s}. \tag{6.6}$$

Apart from a scale factor, d_z and d_z^* are derivatives normal to the surface, and $d_z^* \dot{\zeta}$, the source term for ϕ in (6.3), is a normal dilation rate. The 'Poisson's equation' for ϕ is to be solved separately in each layer under the assumption that $\dot{\zeta} = 0$ implies $\phi = 0$, that is, for the boundary condition $\phi \to 0$ at $|\mathbf{x}| \to \infty$. Formally, the unique solution is

$$\phi = -\nabla_*^{-2} d_z^* \dot{\zeta}, \tag{6.7}$$

∇_*^{-2} being the Green function associated with ∇_*^2. For moderate isopycnal distortion ∇_*^{-2} can be developed in an infinite series around the undistorted Green function

$$\nabla^{-2} \equiv \frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}'| \tag{6.8}$$

in powers of the distortion terms

$$\nabla_*^2 - \nabla^2 = \nabla . (\partial_z \zeta) \nabla - \nabla . \{\eta(\nabla\zeta)(\nabla\zeta) . \nabla\}, \tag{6.9}$$

as

$$\nabla_*^{-2} = \nabla^{-2} - \nabla^{-2}(\nabla_*^2 - \nabla^2)\nabla^{-2} + \dots . \tag{6.10}$$

This is a linear non-local functional in which the leading distortion coefficients are $\partial_z \zeta$ and $\nabla\zeta$, the latter appearing at second order.

All the quantities forming the Lagrangian, including $\tilde{\mathbf{u}} = \nabla\phi$ and $w = \dot{\zeta} + \mathbf{u} . \nabla\zeta$, are now expressible in terms of $\dot{\zeta}$ and ζ. In particular, the canonical momentum (5.7), rewritten with the aid of

$$jw = \eta(\dot{\zeta} + \tilde{\mathbf{u}} . \nabla\zeta) \tag{6.11}$$

(see (3.17), (6.2) and (6.5a)) in the form

$$\pi = \eta\rho\dot{\zeta} - d_z(\rho\phi), \tag{6.12}$$

can be given as

$$\pi = \eta\rho\dot{\zeta} + d_z \rho \nabla_*^{-2} d_z^* \dot{\zeta}. \tag{6.13}$$

Being a homogeneous quadratic functional of $\dot{\zeta}$ with the variational derivative π, the internal-wave Lagrangian must have the form

$$L = \frac{1}{2} \int (\dot{\zeta}\pi - \rho N^2 \zeta^2)\, dx\, dz_0, \tag{6.14}$$

a form which is actually symmetric in $\dot{\zeta}$ because ∇_*^{-2} is a self-adjoint operator which commutes with ρ, a layer constant.

Lagrange's equation of motion (5.13), together with (6.13) above, completely specifies the dynamics. The usual linearized approximation can be recovered at once by setting the right-hand side of (5.13) equal to zero and by setting $\eta = 1, d_z = d_z^* = \partial_z$, and $\nabla_*^{-2} = \nabla^{-2}$ in (6.13), i.e.

$$\pi = \rho\dot{\zeta} + \partial_z \rho \nabla^{-2} \partial_z \dot{\zeta} + O(\nabla\zeta, \partial_z\zeta). \tag{6.15}$$

A substitution of the time derivative of this equation into (5.13), followed by application of ∇^2, yields, to first order,

$$[\nabla^2 + (\partial_z - b)\,\partial_z]\ddot{\zeta} + N^2\nabla^2\zeta \simeq 0. \tag{6.16}$$

The quantity $b(z_0)$ appearing above is the inverse Boussinesq length scale defined previously in (5.14). Although the Boussinesq approximation, $b = 0$, simplifies the equation slightly it is not essential in isopycnal co-ordinates because b is a layered parameter independent of ζ.

7. Hamilton's equations for internal waves

At nonlinear order $\dot{\pi}$ becomes a complicated mixture involving $\ddot{\zeta}$, $\nabla\zeta$, and so on, and it is preferable to solve for $\dot{\zeta}$ as a functional of π and ζ. Formally this is accomplished by Hamilton's equations,

$$\dot{\pi} = -\frac{\delta H}{\delta \zeta}, \quad \dot{\zeta} = \frac{\delta H}{\delta \pi}, \tag{7.1 a, b}$$

with

$$H \equiv \int (\dot{\zeta}\pi)\, dx\, dz_0 - L$$

$$= \frac{1}{2}\int (\dot{\zeta}\pi + \rho N^2 \zeta^2)\, dx\, dz_0, \tag{7.2}$$

but in practice the Hamiltonian functional H is most conveniently constructed by deriving $\dot{\zeta}(\zeta, \pi)$ first. This can be done by starting with (6.12) in the form

$$\dot{\zeta} = \eta^{-1}\rho^{-1}[\pi + d_z(\rho\phi)], \tag{7.3}$$

differentiating by d_z^*, and equating the result to minus $\nabla_*^2\,\phi$:

$$[\nabla_*^2 + d_z^*\eta^{-1}(d_z - b)]\,\phi + d_z^*(\pi/\rho\eta) = 0, \tag{7.4}$$

or

$$\partial_*^2\,\phi + d_z^*(\pi/\rho\eta) = 0 \tag{7.5}$$

for short. The normal gradient of $\pi/\rho\eta$ is the source term in a three-dimensional Poisson-like equation for ϕ in which ∂_*^2 is a distorted Laplacian. This equation defines ϕ uniquely once the boundary conditions at $z_0 = z_b$, z_s are established; for internal-wave dynamics we take these to be rigid-wall conditions, $\zeta = 0$ and $w = 0$, and we write the solution of (7.5) incorporating the boundary conditions symbolically as

$$\phi = -\partial_*^{-2} d_z^*(\pi/\rho\eta), \tag{7.6}$$

with ∂_*^{-2} representing the appropriate integral Green operator. Again, ∂_*^{-2} can be developed in an infinite series starting with the undistorted Green function ∂^{-2},

$$\partial_*^{-2} = \partial^{-2} - \partial^{-2}(\partial_*^2 - \partial^2)\,\partial^{-2} + \dots, \tag{7.7}$$

where

$$\partial^2 = \nabla^2 + \partial_z(\partial_z - b). \tag{7.8}$$

Note that the Boussinesq approximation $b = 0$ converts ∂^2 to the ordinary three-dimensional Laplacian. The required expression for $\dot{\zeta}$ in terms of ζ and π is now, from (7.3),

$$\dot{\zeta} = [1-\eta^{-1}(d_z-b)\,\partial_*^{-2}\,d_z^*]\frac{\pi}{\rho\eta}. \tag{7.9}$$

This equation and (5.13) are a canonical pair describing nonlinear internal waves. They are exact and closed, although (7.9) contains an integral operator ∂_*^{-2} that can be represented in its nonlinear features only as a series expansion in the isopycnal strain components $(\partial_z\zeta, \nabla\zeta)$. The corresponding Hamiltonian functional

$$H[\zeta,\pi] = \frac{1}{2}\int\left[\frac{\pi^2}{\rho\eta}-\pi\eta^{-1}(d_z-b)\,\partial_*^{-2}\,d_z^*\,\eta^{-1}\rho^{-1}\pi+\rho N^2\zeta^2\right]dx\,dz_0, \tag{7.10}$$

is quadratic in π, but in its dependence on $(\partial_z\zeta, \nabla\zeta)$ is likewise expressible only as a series expansion. Incidentally, the operator expression

$$K \equiv [\eta^{-1}-\eta^{-1}(d_z-b)\,\partial_*^{-2}\,d_z^*\,\eta^{-1}]\,\rho^{-1} \tag{7.11}$$

appearing in the Hamiltonian above and forming the kinetic energy

$$T = \frac{1}{2}\int(\pi K\pi)\,dx\,dz_0, \tag{7.12}$$

is actually self-adjoint despite its appearance, as can be shown from the commutation properties of d_z and d_z^* with ρ.

One virtue of the Hamiltonian formulation is that the nonlinear terms in both field equations arise from a single functional H. In the study of weakly nonlinear dynamics this allows the field equations to be truncated unambiguously at the same order without compromise of their canonical form. The series expansion

$$H = {}_0H+{}_1H+\dots \tag{7.13}$$

originates in the expansion of the operator K around its undistorted form,

$$K = K_0+K_1+\dots\ , \tag{7.14}$$

with all of the entities η, d_z, and ∂_*^{-2} contributing terms of various orders of $\partial_z\zeta$ and $\nabla\zeta$. The quadratic Hamiltonian, responsible for the linearized dynamics, is

$${}_0H[\zeta,\pi] = \frac{1}{2}\int\left\{\pi[1-(\partial_z-b)\,\partial^{-2}\,\partial_z]\frac{\pi}{\rho}+\rho N^2\zeta^2\right\}dx\,dz_0, \tag{7.15}$$

while the cubic term, inducing the leading nonlinear correction, is

$${}_1H[\zeta,\pi] = \int[\tfrac{1}{2}(\partial_z\zeta)\,(w_0^2-u_0^2)+(\nabla\zeta).\,w_0\mathbf{u}_0]\,\rho\,dx\,dz_0. \tag{7.16}$$

Here $\mathbf{u}_0$ and w_0 are formal abbreviations for the linear field quantities

$$\mathbf{u}_0 \equiv \nabla\phi_0, \quad w_0 \equiv \frac{\pi}{\rho}+(\partial_z-b)\,\phi_0, \tag{7.17\,a,b}$$

with

$$\phi_0 \equiv -\partial^{-2}\,\partial_z\left(\frac{\pi}{\rho}\right). \tag{7.17\,c}$$

It is easily verified that functional differentiation of ${}_1H$ reproduces the leading non-linear terms in both field equations (5.13), (7.9). We are therefore entitled to conclude that the isopycnal strain tensor $\partial\zeta \equiv (\partial_z\zeta, \nabla\zeta)$ is the scaling parameter for the non-linearity of the dynamics. As a corollary, the linearized equations, when applied to isopycnal co-ordinates, that is, to elevation ζ relative to equilibrium as a function of equilibrium height z_0, are valid for $|\partial\zeta| \ll 1$, *whatever* the ratio of ζ to other length scales describing the medium.

8. Conservation laws

In the canonical formalism a particular conservation law is usually associated with an invariance, or symmetry, of the Hamiltonian (Hill 1951). For internal waves the obvious symmetries are the arbitrariness of time origin and horizontal co-ordinate origin, and these imply conservation of energy and horizontal momentum. Hamilton's principle yields the conserved quantities directly,

$$\delta\int_{t_1}^{t_2} L\,dt = \int \pi\,\delta\zeta\,dx\,dz_0\Big|_{t_1}^{t_2} + (L\,\delta t)\Big|_{t_1}^{t_2}, \tag{8.1}$$

when written as above to account for non-vanishing variations $\delta\zeta$ at $t = t_1, t_2$ and for changes in the temporal limits of integration. Under small arbitrary variations δt, $\delta\mathbf{x}$ in the time and co-ordinate origins, the left-hand side vanishes, while the right-hand side, dependent on the apparent variations

$$\delta\zeta = \dot{\zeta}\,\delta t + \nabla\zeta\,.\,\delta\mathbf{x}, \tag{8.2a}$$

$$\delta t_{1,2} = -\delta t, \tag{8.2b}$$

takes the form

$$O = -\left(H\,\delta t + \delta\mathbf{x}\,.\int \pi\nabla\zeta\,dx\,dz_0\right)\Big|_{t_1}^{t_2}. \tag{8.3}$$

For ideal internal waves the conserved quantity H is the ordinary total energy. If the present formalism can be extended to encompass waves in an ambient shear flow (see the comments in § 9) H will undoubtedly have a more subtle interpretation, because it will continue to be conserved in stable or unstable flows.

The connection between the canonical momentum density $-\pi\nabla\zeta$ and the ordinary mass-transport density $j\rho\mathbf{u}$ can be derived with the help of the relations

$$-\pi\nabla\zeta \equiv (\partial_z\rho\phi - j\rho w)\,\nabla\zeta, \tag{8.4}$$

$$w\nabla\zeta = \nabla\phi - \mathbf{u} \tag{8.5}$$

as

$$j\rho\mathbf{u} = -\pi\nabla\zeta - \partial_z(\rho\phi\nabla\zeta) + \nabla(j\rho\phi). \tag{8.6}$$

Of the three terms contributing to the instantaneous mass transport, the first two are quadratic in the field quantities and account for whatever 'Stokes' streaming may be present at second order in the wave amplitudes. The second term vanishes upon vertical integration, unless the upper surface is free, in which case the surface quantity $-(\rho\phi\nabla\zeta)_s$ remains, while an equivalent term

$$-\int \pi_s\nabla\zeta_s\,dx \tag{8.7}$$

also appears in (8.3) to account for surface-wave momentum (see (5.6)–(5.8)). The second term can therefore be associated with the Stokes streaming generated by surface waves. The third term is principally first-order in the fields, and can be identified with the instantaneous orbital flow; it vanishes upon horizontal integration, producing no net mass transport. All of the mass transport is therefore accounted for by the conserved sum of internal and surface-wave momenta,

$$\int j\rho\mathbf{u}\,dx\,dz_0 = -\int \pi\nabla\zeta\,dx\,dz_0 - \int \pi_s\nabla\zeta_s\,dx. \tag{8.8}$$

Hamilton's principle can be used in combination with an infinitesimal change of amplitude,

$$\delta\zeta = \epsilon\zeta, \quad \delta\pi = \epsilon\pi, \tag{8.9}$$

to extract a relation that is not a strict conservation law, but rather a 'virial theorem' for time-average quantities. Because the nth term in the expanded Hamiltonian is a homogenous $(n+2)$-power functional of π and ζ (or $\partial\zeta$) the left-hand side of

$$\delta\int_{t_1}^{t_2}(T-V)\,dt = \int \pi\,\delta\zeta\,dx\,dz_0\Big|_{t_1}^{t_2} \tag{8.10}$$

has the form

$$\epsilon\int_{t_1}^{t_2}\left[\sum_{n=0}^{\infty}(n+2)\,({}_nT)-2V\right]dt. \tag{8.11}$$

Under the assumption that the right-hand side of (8.10) remains finite, dividing (8.11) by t_2-t_1 yields the limit, as $t_2 \to \infty$,

$$2\langle T-V\rangle + \sum_{n=1}^{\infty} n\langle {}_nT\rangle = 0, \tag{8.12}$$

where $\langle\ \rangle$ denotes time averages. The mean kinetic and potential energies are therefore equal in the linearized approximation, as expected, while for weakly nonlinear motion they differ by approximately $\frac{1}{2}\langle {}_1T\rangle$. In absolute value this term is bounded by

$$|{}_1T| \leqslant 3T\,|\partial\zeta| \leqslant 3(T+V)\,|\partial\zeta|, \tag{8.13}$$

where $|\partial\zeta|^2 = (\nabla\zeta)^2 + (\partial_z\zeta)^2$ (see (7.16)), so that

$$|\langle T-V\rangle| \leqslant \tfrac{3}{2}(T+V)\langle|\partial\zeta|\rangle; \tag{8.14}$$

i.e. in weakly nonlinear motion the average absolute strain $|\partial\zeta|$, times $\frac{3}{2}$, is an upper bound to the relative mean difference of kinetic and potential energy.

9. Discussion

In the foregoing treatment of internal-wave dynamics several interesting regimes of stratified flow have been excluded: inertial waves and free isopycnal circulating flow, for fundamental reasons; and stratified shear flow coupled to, but not generated by, isopycnal motion, for reasons of convenience. Nevertheless, in the restricted regime of ideal internal waves, the canonical formulation has achieved the following new results: nonlinear equations in one degree of freedom that are closed in terms of the fields and associated operators, and a format in which the linearized equations are good for arbitrary amplitude so long as the displacement gradients are small.

The vanishing of the layered potential ϕ at infinity is not required by the canonical approach, and in principle an arbitrary term ϕ' satisfying $\nabla_*^2 \phi' = 0$ could be superposed on each layer. The added current $\tilde{\mathbf{u}}' = \nabla\phi'$ would approach a vector constant $\mathbf{u}_\infty(z_0)$ at large distances from the region disturbed by isopycnal motion, and would therefore represent a background stratified shear flow. The assumption $\mathbf{u}_\infty = 0$ is a particular *ad hoc* assignment of values, necessary only to keep ϕ' bounded. However, when non-zero values of $\mathbf{u}_\infty$ are allowed, the definition of a co-ordinate system stationary with respect to the undisturbed medium is no longer meaningful. Very likely, in a description of the dynamics of stratified shear flow, this arbitrariness will have to be taken into account in the definitions of the variables ζ, ϕ and π; that is, a Galilean-invariant version of the present formalism will have to be devised.

REFERENCES

BRETHERTON, F. P. & GARRET, C. J. R. 1969 *Proc. R. Soc. Lond.* A **302**, 529.

BRETHERTON, F. P. 1970 A note on Hamilton's principle for perfect fluids. *J. Fluid Mech.* **44**, 19.

HENYEY, F. S. 1981 Hamiltonian description of stratified fluid dynamics. *La Jolla Institute Rep.* LJI–R–81–162 (to be published.)

HILL, E. L. 1951 Hamilton's principle and the conservation theorems of mathematical physics. *Rev. Mod. Phys.* **23**, 253.

LANCZOS, C. 1966 *The Variational Principles of Mechanics*. University of Toronto Press.

MEISS, J. D., POMPHREY, N. & WATSON, K. M. 1979 Numerical analysis of weakly nonlinear wave turbulence. *Proc. Nat. Acad. Sci. U.S.A.* **76**, 2109.

MILES, J. W. 1981 Hamiltonian formulations for surface waves. *Appl. Sci. Res.* **37**, 103.